**Many-Body Harmonic Oscillators again**

Just gonna throw out some different ways to solve the same problem.

**Other way**

The last way we did was using canonical transformations each step of the way. This time I’m going to not be particular about that and see what we get. It’ll look more like what we do in QFT. So start with.



where **x**(**R**) and **p**(**R**) denote the displacement and momentum of the oscillator at position **R**, and satisfy,



To find the eigenstates of H we make the substitutions (suppressing the Greek index for now)



Note that periodic boundary conditions restrict the q’s to N distinct values. For instance, if we’re dealing with a paralleliped with side lengths Lx­ = Nxax, Ly = Nyay, Lz = Nzaz (where ai are the lattice spacings), then periodic boundary conditions require:



We’ll choose the Ni distinct values of qi to be:



We can invert the ‘transform’ to get:



Note that the Hermiticity of xR and pR imply:



since for instance,



The commutation relations of the xq’s with the pq’s is:



Now plug this substitution into H:



Define:



Then we have:



Wanna note two properties of Kαβ(q). See the Classical Mechanics folder for justifications on this. But we have:



And second, due to inversion (parity) symmetry, we have:



And for fixed q, Kαβ(q) is a real symmetric 3×3 matrix with non-negative eigenvalues that we denote MionΩqλ2, and real eigenvectors εqλ. Given the two equalities above, we have: Kαβ(q) = Kαβ(-q), and so can expect that a similar relationship holds for the eigenvalues: Ω**q**λ = Ω-**q**,λ. And also for the eigenvectors. But, we can also make the following actually more convenient choice,



which admits the possibility **ε**qλ = **q**, which is useful for longitudinal vibrations. Now make an eigenvector expansion:



Then our commutation relations boil down to:



and our H comes to:



Finally, we can diagonalize H by defining,



where these creation/annihilation operators obey their typical commutation relations,



As we can verify these definitions satisfy the xqλ, pqλ commutation relations:



and now we’ll plug these into our H:



Now can change q → -q in those -q creation/annihilation operators, taking advantage of the evenness of Ωq, to give us:



And then finally we have,



Thus the excitation spectrum of the harmonic lattice can be described as a set of non-interacting bosonic excitations or phonons, having energies Ωqλ created and destroyed by aqλ†,

aqλ respectively. If need be, we can recover the ionic displacements and momenta as



and:



*These results are similar to what we have below, using QFT approaches (so I’ll call this the QFT phase convention).* We can get the previous CMT result by doing a phase transformation of sorts on the a’s. Define b = -ia. Then the new b, b† would still obey canonical commutation relations. Now take our present result and multiply/divide by i to get:



and in particular we may say:



Can do similar manipulations with **p**(**R**) to come to:



which also matches our previous CMT result.